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On the solutions of the second heavenly and Pavlov equations

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Abstract
We have recently solved the inverse scattering problem for one-parameter families of vector fields, and used this result to construct the formal solution of the Cauchy problem for a class of integrable nonlinear partial differential equations connected with the commutation of multidimensional vector fields, such as the heavenly equation of Plebanski, the dispersionless Kadomtsev–Petviashvili (dKP) equation and the two-dimensional dispersionless Toda (2ddT) equation, as well as with the commutation of one-dimensional vector fields, such as the Pavlov equation. We also showed that the associated Riemann–Hilbert inverse problems are powerful tools to establish if the solutions of the Cauchy problem break at finite time, to construct their long-time behaviour and characterize classes of implicit solutions. In this paper, using the above theory, we concentrate on the heavenly and Pavlov equations, (i) establishing that their localized solutions evolve without breaking, unlike the cases of dKP and 2ddT; (ii) constructing the long-time behaviour of the solutions of their Cauchy problems; (iii) characterizing a distinguished class of implicit solutions of the heavenly equation.

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1. Introduction

It was observed long ago [1] that the commutation of multidimensional vector fields can generate integrable nonlinear partial differential equations (PDEs) in arbitrary dimensions. Some of these equations are dispersionless (or quasi-classical) limits of integrable PDEs, having the dispersionless Kadomtsev–Petviashvili (dKP) equation [2, 3] as universal prototype example; they arise in various problems of mathematical physics and are intensively studied in the recent literature (see, for instance, [4–21]). In particular, an elegant integration scheme

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applicable, in general, to nonlinear PDEs associated with Hamiltonian vector fields, was presented in [6] and a nonlinear $\bar{\partial}$-dressing was developed in [12]. Special classes of nontrivial solutions were also derived (see, for instance, [11, 14]).

Distinguished examples of PDEs arising as the commutation conditions $[\hat{L}_1(\lambda), \hat{L}_2(\lambda)] = 0$ of pairs of one-parameter families of vector fields, $\lambda \in \mathbb{C}$ being the spectral parameter, are the following.

(1) The vector nonlinear PDE in $N+4$ dimensions [22]

$$\vec{U}_{t_1 z_2} - \vec{U}_{t_2 z_1} + \left(\vec{U}_{z_1} \cdot \nabla_{\vec{x}}\right)\vec{U}_{z_2} - \left(\vec{U}_{z_2} \cdot \nabla_{\vec{x}}\right)\vec{U}_{z_1} = \vec{0}, \tag{1}$$

where $\vec{U}(t_1, t_2, z_1, z_2, \vec{x}) \in \mathbb{R}^N, \vec{x} = (x^1, \ldots, x^N) \in \mathbb{R}^N$ and $\nabla_{\vec{x}} = (\partial_{x_1}, \ldots, \partial_{x_N})$, associated with the following pair of $(N+1)$ dimensional vector fields:

$$\hat{L}_i = \partial_{t_i} + \lambda\partial_{z_i} + \vec{U}_{z_i} \cdot \nabla_{\vec{x}}, \qquad i = 1, 2. \tag{2}$$

(2) Its dimensional reduction, for $N = 2$ [22]

$$\begin{aligned} &\vec{U}_{tx} - \vec{U}_{zy} + (\vec{U}_y \cdot \nabla_{\vec{x}})\vec{U}_x - (\vec{U}_x \cdot \nabla_{\vec{x}})\vec{U}_y = \vec{0}, \\ &\vec{U} \in \mathbb{R}^2, \qquad \vec{x} = (x, y), \qquad \nabla_{\vec{x}} = (\partial_x, \partial_y), \end{aligned} \tag{3}$$

obtained by renaming the independent variables as follows: $t_1 = z, t_2 = t, x_1 = x, x_2 = y$, associated with the two-dimensional vector fields

$$\hat{L}_1 = \partial_z + \lambda\partial_x + \vec{U}_x \cdot \nabla_{\vec{x}}, \qquad \hat{L}_2 = \partial_t + \lambda\partial_y + \vec{U}_y \cdot \nabla_{\vec{x}}. \tag{4}$$

(3) The divergenceless reduction $\nabla_{\vec{x}} \cdot \vec{U} = 0$ of (3), the celebrated second heavenly equation of Plebanski [23]

$$\theta_{tx} - \theta_{zy} + \theta_{xx}\theta_{yy} - \theta_{xy}^2 = 0, \qquad \theta = \theta(x, y, z, t) \in \mathbb{R}, \qquad x, y, z, t \in \mathbb{R}, \tag{5}$$

describing self-dual vacuum solutions of the Einstein equations, associated with the following pair of Hamiltonian two-dimensional vector fields:

$$\hat{L}_1 \equiv \partial_z + \lambda\partial_x + \theta_{xy}\partial_x - \theta_{xx}\partial_y, \qquad \hat{L}_2 \equiv \partial_t + \lambda\partial_y + \theta_{yy}\partial_x - \theta_{xy}\partial_y. \tag{6}$$

(4) The following system of two nonlinear PDEs in $2+1$ dimensions [24]:

$$\begin{aligned} &u_{xt} + u_{yy} + (uu_x)_x + v_x u_{xy} - v_y u_{xx} = 0, \\ &v_{xt} + v_{yy} + u v_{xx} + v_x v_{xy} - v_y v_{xx} = 0, \end{aligned} \tag{7}$$

arising from the commutation of the two-dimensional vector fields [24]

$$\begin{aligned} &\tilde{L}_1 \equiv \partial_y + (\lambda + v_x)\partial_x - u_x\partial_\lambda, \\ &\tilde{L}_2 \equiv \partial_t + (\lambda^2 + \lambda v_x + u - v_y)\partial_x + (-\lambda u_x + u_y)\partial_\lambda, \end{aligned} \tag{8}$$

and describing a general integrable Einstein–Weyl metric [25].

(5) The $v = 0$ reduction of (7), the celebrated dKP equation

$$(u_t + uu_x)_x + u_{yy} = 0, \qquad u = u(x, y, t) \in \mathbb{R}, \qquad x, y, t \in \mathbb{R}, \tag{9}$$

(the x-dispersionless limit of the celebrated Kadomtsev–Petviashvili equation [26]), associated with the following pair of Hamiltonian two-dimensional vector fields [5, 6]:

$$\hat{L}_1 \equiv \partial_y + \lambda\partial_x - u_x\partial_\lambda, \qquad \hat{L}_2 \equiv \partial_t + (\lambda^2 + u)\partial_x + (-\lambda u_x + u_y)\partial_\lambda, \tag{10}$$

describing the evolution of small amplitude, nearly one-dimensional waves in shallow water [27] near the shore (when the x-dispersion can be neglected), as well as unsteady motion in transonic flow [2] and nonlinear acoustics of confined beams [3].

(6) The $u = 0$ reduction of (7), the Pavlov equation [15]

$$v_{xt} + v_{yy} = v_y v_{xx} - v_x v_{xy}, \qquad v = v(x, y, t) \in \mathbb{R}, \qquad x, y, t \in \mathbb{R}, \tag{11}$$

associated with the non-Hamiltonian one-dimensional vector fields [16]

$$\hat{L}_1 \equiv \partial_y + (\lambda + v_x)\partial_x, \qquad \hat{L}_2 \equiv \partial_t + (\lambda^2 + \lambda v_x - v_y)\partial_x. \tag{12}$$

(7) The two-dimensional dispersionless Toda (2ddT) equation [28, 29]

$$\phi_{\zeta_1\zeta_2} = (\mathrm{e}^{\phi_t})_t, \qquad \phi = \phi(\zeta_1, \zeta_2, t) \tag{13}$$

(or $\varphi_{\zeta_1\zeta_2} = (\mathrm{e}^{\varphi})_{tt}$, $\varphi = \phi_t$), associated with the pair of Hamiltonian vector fields [7]

$$\begin{aligned} \hat{L}_1 &= \partial_{\zeta_1} + \lambda \mathrm{e}^{\frac{\phi_t}{2}} \partial_t + \left(-\lambda\left(\mathrm{e}^{\frac{\phi_t}{2}}\right)_t + \frac{\phi_{\zeta_1 t}}{2}\right)\lambda\partial_\lambda, \\ \hat{L}_2 &= \partial_{\zeta_2} + \lambda^{-1}\mathrm{e}^{\frac{\phi_t}{2}} \partial_t + \left(\lambda^{-1}\left(\mathrm{e}^{\frac{\phi_t}{2}}\right)_t - \frac{\phi_{\zeta_2 t}}{2}\right)\lambda\partial_\lambda, \end{aligned} \tag{14}$$

describing integrable heavens [30, 31] and Einstein–Weyl geometries [32–34]; whose string equations solutions [8] are relevant in the ideal Hele–Shaw problem [35–39].

The inverse spectral transform (IST) for one-parameter families of multidimensional vector fields, developed in [22] (see also [40]), has allowed one to construct the formal solution of the Cauchy problem for the nonlinear PDEs (3) and (5) in [22], for equations (7) and (9) in [24], for equation (11) in [41] and for the wave form $(\mathrm{e}^{\phi_t})_t = \phi_{xx} + \phi_{yy}$ of equation (13) in [42]. This IST, introducing interesting novelties with respect to the classical IST for soliton equations [27, 43], turns out to be, together with its associated nonlinear Riemann–Hilbert (RH) dressing, an efficient tool to study several properties of the solution space of the PDE under consideration: (i) the characterization of a distinguished class of spectral data for which the associated nonlinear RH problem is linearized and solved, corresponding to a class of implicit solutions of the PDE (for the dKP and 2ddT equations respectively in [45] and in [42], and for the Dunajski generalization [16] of the heavenly equation in [44]); (ii) the construction of the long-time behaviour of the solutions of the Cauchy problem (for the dKP and 2ddT equations respectively in [45] and [42]); (iii) the possibility of establishing whether or not the lack of dispersive terms in the nonlinear PDE causes the breaking of localized initial profiles (for the dKP and 2ddT equations respectively in [45] and in [42]) and, if yes, to investigate in a surprisingly explicit way the analytic aspects of such a wave breaking, as was done for the dKP equation in [45].

In this paper we first develop, in section 2, the RH dressing schemes for equations (5) and (11). Then we use these two schemes

(i) To show that, in contrast to the cases of the dKP and 2ddT equations, localized solutions of the PDEs (5) and (11) do not break (in section 2);
(ii) To construct the long-time behaviour of the solutions of their Cauchy problems (in section 3);
(iii) To characterize a distinguished class of spectral data for which the nonlinear RH problem of equation (5) is linearized and solved, corresponding to a distinguished class of implicit solutions of (5) (in section 4).

2. Riemann–Hilbert dressing

In this section we present the RH dressing schemes for equations (5) and (11), and show that localized solutions of these two models do not break. Both schemes can be extracted from the inverse spectral transforms presented in [22] and in [41]. In solving the Cauchy problems for equations (5) and (11), the RH data of these dressing schemes are connected to the initial conditions via the direct spectral transforms presented in [22] and in [41].

(1) *RH dressing for* (5). Consider the vector nonlinear RH problem on the real line:

$$\vec{\pi}^+(\lambda) = \vec{\pi}^-(\lambda) + \vec{R}(\vec{\pi}^-(\lambda), \lambda), \qquad \lambda \in \mathbb{R}, \tag{15}$$

where $\vec{\pi}^+(\lambda), \vec{\pi}^-(\lambda) \in \mathbb{C}^2$ are two-dimensional vector functions analytic in the upper and lower halves of the complex λ plane, normalized as follows:

$$\begin{aligned} &\vec{\pi}^\pm(\lambda) = \vec{v}(\lambda; x, y, z, t) + O(\lambda^{-1}), \qquad |\lambda| \gg 1, \\ &\vec{v}(\lambda; x, y, z, t) = \begin{pmatrix} x - \lambda z \\ y - \lambda t \end{pmatrix}, \end{aligned} \tag{16}$$

and the spectral data $\vec{R}(\vec{\zeta}, \lambda) = (R_1(\zeta_1, \zeta_2, \lambda), R_2(\zeta_1, \zeta_2, \lambda)) \in \mathbb{C}^2$, defined for $\vec{\zeta} \in \mathbb{C}^2$, $\lambda \in \mathbb{R}$, satisfy the following properties:

$$\vec{\mathcal{R}}(\overline{\vec{\mathcal{R}}(\overline{\vec{\zeta}}, \lambda)}, \lambda) = \vec{\zeta}, \qquad \forall \vec{\zeta} \in \mathbb{C}^2, \quad \text{reality constraint}, \tag{17a}$$

$$\{\mathcal{R}_1, \mathcal{R}_2\}_{\vec{\zeta}} = 1, \quad \text{heavenly constraint}, \tag{17b}$$

where $\vec{\mathcal{R}}(\vec{\zeta}, \lambda) := \vec{\zeta} + \vec{R}(\vec{\zeta}, \lambda)$ and $\{\cdot, \cdot\}_{\vec{\zeta}}$ is the usual Poisson bracket with respect to the variables $\vec{\zeta} = (\zeta_1, \zeta_2)$. Then, assuming uniqueness of the solution of such a RH problem and of its linearized version, it follows that $\vec{\pi}^\pm$ are solutions of the linear problems $\hat{L}_{1,2}\vec{\pi}^\pm = \vec{0}$, where $\hat{L}_{1,2}$ are defined in (6), and

$$\begin{pmatrix} \theta_y \\ -\theta_x \end{pmatrix} = \vec{F}(x, y, z, t) \in \mathbb{R}^2 \tag{18}$$

is the solution of the heavenly equation (5), where

$$\vec{F}(x, y, z, t) = \int_{\mathbb{R}} \frac{d\lambda}{2\pi i} \vec{R}(\pi_1^-(\lambda; x, y, z, t), \pi_2^-(\lambda; x, y, z, t), \lambda). \tag{19}$$

As a consequence of equations $\hat{L}_{1,2}\vec{\pi}^\pm = \vec{0}$, it follows that, for $|\lambda| \gg 1$

$$\begin{aligned} &\pi_1^\pm = x - \lambda z - \theta_y \lambda^{-1} + \theta_t \lambda^{-2} + O(\lambda^{-3}), \\ &\pi_2^\pm = y - \lambda t + \theta_x \lambda^{-1} - \theta_z \lambda^{-2} + O(\lambda^{-3}). \end{aligned} \tag{20}$$

Proof. We apply the operators $\hat{L}_{1,2}$ in (4) to the RH problem (15), where

$$\vec{U} = -\lim_{\lambda\to\infty} \lambda \left(\vec{\pi}^\pm - \begin{pmatrix} x - \lambda z \\ y - \lambda t \end{pmatrix} \right), \tag{21}$$

obtaining the vector equation $\hat{L}_j\vec{\pi}^+ = J(\vec{\pi}^-, \lambda)\hat{L}_j\vec{\pi}^-$, $j = 1, 2$, where J is the Jacobian matrix of the transformation (15): $J_{kl}(\vec{\zeta}, \lambda) = \partial\mathcal{R}_k(\vec{\zeta}, \lambda)/\partial\zeta_l$. Since, due to (21), $\hat{L}_j\vec{\pi}^\pm \to 0$ as $\lambda \to \infty$, by uniqueness we infer that $\vec{\pi}^\pm$ are eigenfunctions of the vector fields $\hat{L}_j$: $\hat{L}_j\vec{\pi}^\pm = \vec{0}$. Consequently, $\vec{U}$ is a solution of equation (3). In addition, if the reality constraint (17*a*) is imposed, then, by uniqueness, it follows, from (15), that $\overline{\vec{\pi}^+} = \vec{\pi}^-$, implying the reality of $\vec{U}$. Finally, if the heavenly constraint (17*b*) is imposed, then $\{\pi_1^+,\pi_2^+\}=\{\pi_1^-,\pi_2^-\}$. Since $\{\pi_1^\pm,\pi_2^\pm\}\to 1$ as $|\lambda|\to\infty$, the analyticity of the eigenfunctions implies that $\{\pi_1^+,\pi_2^+\}=\{\pi_1^-,\pi_2^-\}=1$. Applying $\hat{L}_{1,2}$ to these two equations, one infers that $\nabla_{\vec{x}}\cdot\vec{U}=0$, implying the existence of a potential θ such that $\vec{U}=(\theta_y,-\theta_x)$, and implying also that the two vector fields $\hat{L}_{1,2}$ are Hamiltonian, with Hamiltonians $(H_1,H_2)=\nabla_{\vec{x}}\theta$, reducing to the vector fields (6). Then system (3) reduces to the heavenly equation (5) and equation (18) follows from (21) and from the integral equations characterizing the RH problem. □

(2) *RH dressing for* (11). Consider the scalar nonlinear RH problem on the real line

$$\pi^+(\lambda)=\pi^-(\lambda)+R(\pi^-(\lambda),\lambda),\qquad \lambda\in\mathbb{R}, \tag{22}$$

where $\pi^+(\lambda),\pi^-(\lambda)\in\mathbb{C}$ are scalar functions analytic in the upper and lower halves of the complex λ plane, normalized as follows:

$$\begin{aligned}&\pi^\pm(\lambda;x,y,t)=\nu(\lambda;x,y,t)+O(\lambda^{-1}),\qquad |\lambda|\gg 1,\\&\nu(\lambda;x,y,t)=-\lambda^2t-\lambda y+x,\end{aligned} \tag{23}$$

and the spectral datum $R(\zeta,\lambda)\in\mathbb{C}$ satisfies the following reality constraint:

$$\mathcal{R}(\overline{\mathcal{R}(\bar{\zeta},\lambda)},\lambda)=\zeta,\qquad \forall\zeta\in\mathbb{C},\quad \lambda\in\mathbb{R}, \tag{24}$$

where $\mathcal{R}(\zeta,\lambda):=\zeta+R(\zeta,\lambda)$. Then, assuming uniqueness of the solution of such a RH problem and of its linearized version, it follows that $\pi^\pm$ are solutions of the linear problems $\hat{L}_{1,2}\pi^\pm=0$, where $\hat{L}_{1,2}$ are defined in (12), and

$$v=F(x,y,t)\in\mathbb{R} \tag{25}$$

is a solution of equation (11), where

$$F(x,y,t)=\int_{\mathbb{R}}\frac{d\lambda}{2\pi i}R(\pi^-(\lambda;x,y,t),\lambda). \tag{26}$$

Proof. We apply the operators $\hat{L}_{1,2}$ in (12) to the RH problem (22), where

$$v(x,y,t)=-\lim_{\lambda\to\infty}\lambda(\pi^\pm+\lambda^2t+\lambda y-x), \tag{27}$$

obtaining $\hat{L}_j\pi^+=J(\pi^-,\lambda)\hat{L}_j\pi^-$, $j=1,2$, where $J(\zeta,\lambda)=\partial\mathcal{R}(\zeta,\lambda)/\partial\zeta$. Since $\hat{L}_1\pi^\pm\to 0$ as $\lambda\to\infty$, it follows that, by uniqueness, $\pi^\pm$ are eigenfunctions of $\hat{L}_1$: $\hat{L}_1\pi^\pm=0$. Consequently, we have the asymptotics

$$\begin{aligned}&\pi^\pm=-\lambda^2t-\lambda y+x-\frac{v}{\lambda}+\frac{\beta}{\lambda^2}+O(\lambda^{-3}),\qquad |\lambda|\gg 1.\\&\beta_x=v_y+v_x^2,\end{aligned} \tag{28}$$

implying that also $\hat{L}_2\pi^\pm\to 0$ as $\lambda\to\infty$. Therefore, again by uniqueness, $\hat{L}_{1,2}\pi^\pm=0$, v satisfies equation (11), and equation (25) follows from (27) and from the integral equations characterizing the RH problem. If, in addition, the reality constraint (24) is imposed, then, by uniqueness, $\overline{\pi^+}=\pi^-$ and, consequently, $v\in\mathbb{R}$. □

We recall that, in the dressing schemes for the dKP and 2ddT equations, the spectral mechanism responsible for breaking is the presence, in the normalizations of the solutions of their RH problems, of the unknown fields, so that the inverse formulae define the solutions of the PDEs only implicitly [42, 45]. Since the normalizations (16), (23) depend only on the spacetime independent variables, such a breaking mechanism is absent here, and we expect that localized solutions of (5) and (11) do not break during their evolution.

3. Long-time behaviour of the solutions

The IST provides an effective tool to construct the long-time behaviour of the solutions of integrable PDEs. In this section we investigate the long-time behaviour of the solutions of the nonlinear RH problems (15), (22), and, consequently, of the solutions of the Cauchy problems for equations (5) and (11).

Asymptotics of equation (5). We study the long-time $t \gg 1$ regime in the space regions

$$x = \tilde{x} + v_1 t, \quad y = v_2 t, \quad z = v_3 t, \quad \tilde{x}, v_1, v_2, v_3 = O(1), \quad v_2, v_3 \neq 0, \quad t \gg 1. \tag{29}$$

The system of nonlinear integral equations characterizing the solutions of the RH problem (15) is conveniently written in the form

$$\phi_j(\lambda) = \frac{1}{2\pi \mathrm{i}} \int_{\mathbb{R}} \frac{\mathrm{d}\lambda'}{\lambda' - (\lambda - \mathrm{i}0)} R_j(\tilde{x} + (v_1 - \lambda' v_3)t + \phi_1(\lambda'), (v_2 - \lambda')t + \phi_2(\lambda'), \lambda'), \quad j = 1, 2, \tag{30}$$

where

$$\phi_1(\lambda) = \pi_1^-(\lambda) - (x - \lambda z), \qquad \phi_2(\lambda) = \pi_2^-(\lambda) - (y - \lambda t). \tag{31}$$

The fast decay, in t, of ϕ_j, $j = 1, 2$, due to the two different linear growths in t of the first two arguments of R_1, R_2, is partially contrasted if $v_1 = v_2 v_3$; i.e., on the saddle surface

$$x = \tilde{x} + \frac{yz}{t} \qquad (v_1 = v_2 v_3). \tag{32}$$

Indeed, on such a two-dimensional manifold, the first two arguments of R_1, R_2 in (30) exhibit the same linear growth in t

$$\phi_j(\lambda) = \frac{1}{2\pi \mathrm{i}} \int_{\mathbb{R}} \frac{\mathrm{d}\lambda'}{\lambda' - (\lambda - \mathrm{i}0)} R_j(\tilde{x} + v_3(v_2 - \lambda')t + \phi_1(\lambda'), (v_2 - \lambda')t + \phi_2(\lambda'), \lambda'), \quad j = 1, 2. \tag{33}$$

Since, in this case, the main contribution to the integral occurs when $\lambda' \sim v_2$, we make the change of variable $\lambda' = v_2 + \mu'/t$, obtaining

$$\phi_j(\lambda) \sim \frac{1}{2\pi \mathrm{i} t} \int_{\mathbb{R}} \frac{\mathrm{d}\mu'}{\mu'/t - (\lambda - v_2 - \mathrm{i}0)} R_j\left(\tilde{x} - v_3\mu' + \phi_1\left(v_2 + \frac{\mu'}{t}\right), -\mu' + \phi_2\left(v_2 + \frac{\mu'}{t}\right), v_2\right), \qquad j = 1, 2. \tag{34}$$

If $|\lambda - v_2| \gg t^{-1}$, equation (34) implies that $\phi_j(\lambda) = O(t^{-1})$, $j = 1, 2$

$$\phi_j(\lambda) \sim -\frac{1}{2\pi \mathrm{i}(\lambda - (v_2 + \mathrm{i}0))t} \int_{\mathbb{R}} \mathrm{d}\mu' R_j\left(\tilde{x} - v_3\mu' + \phi_1\left(v_2 + \frac{\mu'}{t}\right), -\mu' + \phi_2\left(v_2 + \frac{\mu'}{t}\right), v_2\right), \qquad j = 1, 2, \tag{35}$$

while, for $\lambda - v_2 = \mu/t$, $\mu = O(1)$, we have that $\phi_j(\lambda) = O(1)$, $j = 1, 2$

$$\phi_j\left(v_2 + \frac{\mu}{t}\right) \sim \frac{1}{2\pi \mathrm{i}} \int_{\mathbb{R}} \frac{\mathrm{d}\mu'}{\mu' - (\mu - \mathrm{i}0)} R_j\left(\tilde{x} - v_3\mu' + \phi_1\left(v_2 + \frac{\mu'}{t}\right), -\mu' + \phi_2\left(v_2 + \frac{\mu'}{t}\right), v_2\right), \qquad j = 1, 2. \tag{36}$$

Therefore it is not possible to neglect, in the integral equations (34), $\phi_{1,2}$ in the arguments of $R_{1,2}$ and these integral equations remain nonlinear also in the long-time regime. Summarizing, due to equations (36), (34), (18) and (19), the long-time behaviour of the solutions of the Cauchy problem for the heavenly equation on the saddle (32) reads

$$\begin{pmatrix} -\theta_y \\ \theta_x \end{pmatrix} = \frac{1}{t}\vec{F}_\infty\left(x - \frac{yz}{t}, \frac{y}{t}, \frac{z}{t}\right) + o(t^{-1}), \qquad x - \frac{yz}{t} = O(1), \qquad t \gg 1, \tag{37}$$

where

$$\vec{F}_\infty(\tilde{x}, v_2, v_3) = -\frac{1}{2\pi \mathrm{i}}\int_{\mathbb{R}} \mathrm{d}\mu\, \vec{R}(\tilde{x} - v_3\mu + a_1(\mu; \tilde{x}, v_2, v_3), -\mu + a_2(\mu; \tilde{x}, v_2, v_3), v_2). \tag{38}$$

and $a_j(\mu; \tilde{x}, v_2, v_3)$, $j = 1, 2$ are the solutions of the following nonlinear integral equations:

$$a_j(\mu) = \frac{1}{2\pi \mathrm{i}}\int_{\mathbb{R}} \frac{\mathrm{d}\mu'}{\mu' - (\mu - \mathrm{i}0)} R_j(\tilde{x} - v_3\mu' + a_1(\mu'), -\mu' + a_2(\mu'), v_2), \qquad j = 1, 2. \tag{39}$$

Outside the saddle, the solution decays faster than $1/t$.

We remark that the long-time behaviour (37) is formally the same as that of solutions of the linearized heavenly equation $\theta_{tx} - \theta_{zy} = 0$; the nonlinearity manifests only in the fact that $\vec{F}_\infty$ is obtained, from the scattering data, through the solution of the nonlinear integral equations (39). These equations characterize the new RH problem

$$\vec{A}^+(\mu) = \vec{A}^-(\mu) + \vec{R}(\vec{A}^-(\mu), v_2), \qquad \mu \in \mathbb{R}, \tag{40}$$

where $\vec{A}^\pm(\mu; \tilde{x}, v_2, v_3) \in \mathbb{C}^2$ are two-dimensional vector functions analytic in the upper and lower halves of the complex μ plane, normalized as follows:

$$\vec{A}^\pm(\mu) = \begin{pmatrix} \tilde{x} - v_3\mu \\ -\mu \end{pmatrix} + O(\mu^{-1}), \tag{41}$$

$\vec{R}(\vec{\zeta}, \lambda)$ are the heavenly spectral data and $\vec{a}(\mu; \tilde{x}, v_2, v_3) \equiv \vec{A}^-(\mu; \tilde{x}, v_2, v_3) - (\tilde{x} - v_3\mu, -\mu)^T$.

The above analysis extends, with minor modifications, to equation (11), for which we give just the results.

Asymptotics of equation (11). Let $t \gg 1$ and

$$x = \tilde{x} + v_1 t, \qquad y = v_2 t, \qquad \tilde{x}, v_1, v_2 = O(1), \qquad v_2 \neq 0, \quad t \gg 1. \tag{42}$$

On the parabola

$$x = \tilde{x} - \frac{y^2}{4t} \qquad \left(v_1 = -\frac{v_2^2}{4}\right), \tag{43}$$

the analogue of the heavenly saddle (32), the long-time behaviour of the solutions of equation (11) is given by

$$v = \frac{1}{\sqrt{t}}F_\infty\left(x + \frac{y^2}{4t}, \frac{y}{t}\right) + o\left(\frac{1}{\sqrt{t}}\right), \tag{44}$$

where

$$F_\infty(\tilde{x}, v_2) = \frac{1}{2\pi \mathrm{i}}\int_{\mathbb{R}} \mathrm{d}\mu\, R\left(\tilde{x} - \mu^2 + a(\mu; \tilde{x}, v_2), -\frac{v_2}{2}\right), \tag{45}$$

$R(\zeta, \lambda)$ is the spectral datum of equation (11), and $a(\mu; \tilde{x}, v_2)$ is the unique solution of the nonlinear integral equation

$$a(\mu) = \frac{1}{2\pi \mathrm{i}} \int_{\mathbb{R}} \frac{\mathrm{d}\mu'}{\mu' - (\mu - \mathrm{i}0)} R\left(\tilde{x} + \mu'^2 + a(\mu'), \frac{v_2}{2}\right), \tag{46}$$

characterizing the solution of the following scalar nonlinear Riemann problem on the real axis:

$$\begin{aligned} &A^+(\mu) = A^-(\mu) + R\left(A^-(\mu), -\frac{v_2}{2}\right), && \mu \in \mathbb{R}, \\ &A^\pm\left(\mu; \tilde{x}, \frac{v_2}{2}\right) = \tilde{x} - \mu^2 + O(\mu^{-1}), && |\mu| \gg 1, \end{aligned} \tag{47}$$

where $a\left(\mu; \tilde{x}, \frac{v_2}{2}\right) \equiv A^-(\mu) - (\tilde{x} - \mu^2)$. Outside the parabola (43), the solution decays faster than $1/\sqrt{t}$.

4. Implicit solutions of the heavenly equation

In this section we construct a class of explicit solutions of the vector nonlinear RH problem (15)–(17) and, correspondingly, a class of implicit solutions of the heavenly equation (5), parametrized by an arbitrary real function of two variables.

Suppose that the RH data in (15) are given by

$$R_j(\zeta_1, \zeta_2, \lambda) = \zeta_j\left(\mathrm{e}^{\mathrm{i}(-)^{j+1} f(\zeta_1\zeta_2, \lambda)} - 1\right), \qquad j = 1, 2, \tag{48}$$

in terms of the new real spectral function $f(\zeta, \lambda)$ depending on π_1^- and π_2^- only through their product. Then the RH problem (15) becomes

$$\begin{aligned} &\pi_1^+ = \pi_1^- \, \mathrm{e}^{\mathrm{i} f\left(\pi_1^-\pi_2^-, \lambda\right)}, && \lambda \in \mathbb{R}, \\ &\pi_2^+ = \pi_2^- \, \mathrm{e}^{-\mathrm{i} f\left(\pi_1^-\pi_2^-, \lambda\right)} && \end{aligned} \tag{49}$$

and the following properties hold.

(1) The reality and the heavenly constraints (17) are satisfied.

(2) $\pi_1^+\pi_2^+ = \pi_1^-\pi_2^-$. Consequently, $(\pi_1^+\pi_2^+)$ is just a polynomial in λ

$$\pi_1^+\pi_2^+ = \pi_1^-\pi_2^- = zt\lambda^2 - (xt + yz)\lambda + xy - z\theta_x + t\theta_y \equiv w(\lambda), \tag{50}$$

and the vector nonlinear RH problem (15) decouples into two linear scalar RH problems

$$\pi_1^+ = \pi_1^- \, \mathrm{e}^{\mathrm{i} f(w(\lambda), \lambda)}, \qquad \pi_2^+ = \pi_2^- \, \mathrm{e}^{-\mathrm{i} f(w(\lambda), \lambda)}. \tag{51}$$

(3) Since, from (51),

$$\pi_j^+ \, \mathrm{e}^{\mathrm{i}(-)^j f^+(\lambda)} = \pi_j^- \, \mathrm{e}^{\mathrm{i}(-)^j f^-(\lambda)}, \qquad j = 1, 2, \tag{52}$$

where

$$f^\pm(\lambda) = \frac{1}{2\pi \mathrm{i}} \int_{\mathbb{R}} \frac{\mathrm{d}\lambda'}{\lambda' - (\lambda \pm \mathrm{i}0)} f(w(\lambda'), \lambda'), \tag{53}$$

also $\pi_j^+ \, \mathrm{e}^{\mathrm{i}(-)^j f^+(\lambda)}$, $j = 1, 2$ are polynomials in λ. We expand them in powers of λ, for $|\lambda| \gg 1$, and introduce the notation

$$\langle \lambda^n f \rangle = \frac{1}{2\pi} \int_{\mathbb{R}} \lambda^n f(w(\lambda), \lambda) \, \mathrm{d}\lambda, \qquad n \in \mathbb{N}. \tag{54}$$

From the positive power expansions, it follows that:

$$\begin{aligned} &\pi_1^+ \, \mathrm{e}^{-\mathrm{i} f^+(\lambda)} = \pi_1^- \, \mathrm{e}^{-\mathrm{i} f^-(\lambda)} = x - z\lambda - z\langle f \rangle, \\ &\pi_2^+ \, \mathrm{e}^{\mathrm{i} f^+(\lambda)} = \pi_2^- \, \mathrm{e}^{\mathrm{i} f^-(\lambda)} = y - t\lambda + t\langle f \rangle, \end{aligned} \tag{55}$$

implying the following explicit solution of the RH problem (51):

$$\pi_1^{\pm} = (x - \lambda z - z\langle f\rangle)\,\mathrm{e}^{\mathrm{i}f^{\pm}(\lambda)}, \qquad \pi_2^{\pm} = (y - \lambda t + t\langle f\rangle)\,\mathrm{e}^{-\mathrm{i}f^{\pm}(\lambda)}. \tag{56}$$

From the coefficients of the λ^{-1} terms, which must be zero, one finally obtains the two conditions

$$\theta_y = x\langle f\rangle - z\left(\langle\lambda f\rangle + \tfrac{1}{2}\langle f\rangle^2\right), \qquad \theta_x = y\langle f\rangle - t\left(\langle\lambda f\rangle - \tfrac{1}{2}\langle f\rangle^2\right). \tag{57}$$

Since $\langle f\rangle$ and $\langle\lambda f\rangle$ depend, through w, on θ_x and θ_y (see (50)), it follows that (57) is a system of algebraic equations defining implicitly the solution (θ_x, θ_y) of the heavenly equation.

In addition, equating to zero the coefficients of the λ^{-2} terms (using (20)), one obtains

$$\begin{aligned} \theta_t &= \theta_y\langle f\rangle - x\left(\langle\lambda f\rangle + \tfrac{1}{2}\langle f\rangle^2\right) + z\left(\langle\lambda^2 f\rangle + \langle f\rangle\langle\lambda f\rangle + \tfrac{1}{6}\langle f\rangle^3\right), \\ \theta_z &= -\theta_x\langle f\rangle - y\left(\langle\lambda f\rangle - \tfrac{1}{2}\langle f\rangle^2\right) + t\left(\langle\lambda^2 f\rangle - \langle f\rangle\langle\lambda f\rangle + \tfrac{1}{6}\langle f\rangle^3\right), \end{aligned} \tag{58}$$

and manipulating equations (58) and (57), one shows that the solutions of (5) generated by the spectral data (48) satisfy the following linear PDE:

$$t\theta_t - z\theta_z + y\theta_y - x\theta_x = 0. \tag{59}$$

Substituting in (5) the expression of θ_t given in (59), one obtains the lower dimensional nonlinear constraint

$$(x\theta_x)_x - y\theta_{xy} + z\theta_{xz} - t\theta_{yz} + t\left(\theta_{xx}\theta_{yy} - \theta_{xy}^2\right) = 0 \tag{60}$$

satisfied by the solutions of (5) generated by the spectral data (48).

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